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Convergence of Derivatives of Optimal Nodal Splines

Vittoria Demichelis*

Department of Mathematics, University of Turin, Via Carlo Alberto 10, I-10123 Turin, Italy

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Optimal nodal spline interpolants Wf of order m which have local support can be used to interpolate a continuous function f at a set of mesh points. These splines belong to a spline space with simple knots at the mesh points as well as at $m-2$ arbitrary points between any two mesh points and they reproduce polynomials of order m. It has been shown that, for a sequence of locally uniform meshes, these splines converge uniformly for any $f \in \mathbf{C}$ as the mesh norm tends to zero. In this paper, we derive a set of sufficient conditions on the sequence of meshes for the uniform convergence of D^jWf to D^jf for $f \in \mathbf{C}^s$ and $j = 1, ..., s < m$. In addition we give a bound for D^rWf with $s < r < m$. Finally, we use optimal nodal spline interpolants for the numerical evaluation of Cauchy principal value integrals.

1. INTRODUCTION

In the construction of spline approximation operators it is desirable to obtain the three properties of locality, interpolation, and optimal polynomial reproduction. However, it was shown in [4] that, in the case where the knots of the spline space are chosen to coincide with the interpolation points, the two properties of locality and interpolation are incompatible for quadratic or higher order splines.

Employing a procedure based on the introduction of additional knots, De Villiers and Rohwer [4, 5] constructed, for arbitrary order, an optimal nodal spline approximation operator W which was indeed shown to possess these three desired properties. Similar approaches have been followed for quadratic splines in [8] and for arbitrary order splines in [1].

The approximation properties of W were studied in both [3] and [6]. In the present paper we will continue the investigation of De Villiers and Rohwer on how well the nodal spline Wf approximates a smooth function. In Section 2 we shall introduce the nodal splines and report the convergence

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results from [3] and [6]. In Section 3 we give the conditions under which $D^j Wf$ converges uniformly to $D^j f$, for $f \in \mathbf{C}^s$ and $j = 1, ..., s < m$, where m is the order of the spline, and D^j is the jth derivative operator. In addition we give a bound for $D^j Wf$ for $s < j < m$. In Section 4 we give an application of our result by showing the uniform convergence for a sequence of Cauchy principal value (CPV) integrals of optimal nodal spline interpolants. These splines were already used in [10] for product integration of singular integrands and in [2] for numerical evaluation of CPV integrals by subtracting out the singularity.

2. OPTIMAL NODAL SPLINE APPROXIMATION

In this section we give the necessary background material on optimal nodal spline interpolants based on the work in [6].

Let $[a, b]$ be a given finite interval of the real line $\mathbf{R}$. For a fixed integer $m \geqslant 2$, let $n \geqslant m-1$. We define the partition X_n of $[a, b]$ by

$$X_n \mid x_0 := a < x_1 < \cdots < x_{(m-1)n} := b.$$

Setting $\tau_i := x_{(m-1)i}$, $0 \leqslant i \leqslant n$, we define $\Pi_n := \{\tau_i;\, i = 0, ..., n\}$, so that $\Pi_n \subset X_n$. We will denote the points of Π_n and $X_n \backslash \Pi_n$ respectively by primary and by secondary knots corresponding to the partition X_n.

We write $\mathbf{P}^m$ for the set of polynomials of order m (degree $\leqslant m-1$) and $\mathbf{S}_{m,n}$ for the set of polynomial splines of order m with simple knots at the points x_i, $i = 1, ..., (m-1)\, n - 1$, so that $\mathbf{S}_{m,n} \subset \mathbf{C}^{m-2}[a, b]$.

We define

$$r_n := \max_{0 \leqslant i,\, j \leqslant (m-1)\, n-1;\; |i-j| = 1} \frac{x_{i+1} - x_i}{x_{j+1} - x_j}, \tag{1}$$

and

$$R_n := \max_{0 \leqslant i,\, j \leqslant n-1;\; |i-j| = 1} \frac{\tau_{i+1} - \tau_i}{\tau_{j+1} - \tau_j}. \tag{2}$$

We say that the sequence of partitions $\{X_n;\, n = m-1, m, ...\}$ ($\{\Pi_n\}$) is locally uniform if there exists a constant $\bar{\rho}_m \geqslant 1 (\hat{\rho}_m \geqslant 1)$ such that $r_n \leqslant \bar{\rho}_m (R_n \leqslant \hat{\rho}_m)$ for all n. We shall say that a sequence of spline spaces $\{\mathbf{S}_{m,n}\}$ is locally uniform if the sequence of underlying partitions $\{X_n\}$ is locally uniform.

We denote by H_n the norm of the primary partition Π_n

$$H_n := \max_{0 \leqslant i \leqslant n-1} (\tau_{i+1} - \tau_i). \tag{3}$$

In [4, 5] it was proved constructively that there exists, for arbitrary order m, a nodal spline approximation operator $W_n\colon \mathbf{B}[a, b] \to \mathbf{S}_{m,n}$, where $\mathbf{B}[a, b]$ denotes the set of real-valued functions on $[a, b]$, with the following properties:

$$W_n f(\tau_i) = f(\tau_i), \qquad i = 0, 1, ..., n; \tag{4}$$

$$W_n p = p \qquad \text{for all} \quad p \in \mathbf{P}^m; \tag{5}$$

W_n is local in the sense that, for a fixed $x \in [a, b]$, the value of $W_n f$ at x depends on the values of f at at most $(m+1)$ neighbouring primary knots.

In the linear case $m = 2$, $W_n f$ is trivially given by the piecewise linear interpolant of f. Assuming henceforth that $m \geqslant 3$, the defining formula for $W_n f$ on $[a, b]$ is given by

$$W_n f(x) := \sum_{i = p_\mu}^{q_\mu} f(\tau_i) w_i(x), \qquad x \in [\tau_\mu, \tau_{\mu+1}], \quad \mu = 0, 1, ..., n-1; \tag{6}$$

where, following the constructive remark in [2], we define

$$p_\mu := \begin{cases} 0, & \mu = 0, ..., i_1 - 2; \\ \mu - i_1 + 1, & \mu = i_1 - 1, ..., n - i_0; \\ n - (m-1), & \mu = n - i_0 + 1, ..., n-1; \end{cases} \tag{7}$$

$$q_\mu := \begin{cases} m - 1, & \mu = 0, ..., i_1 - 2; \\ \mu + i_0, & \mu = i_1 - 1, ..., n - i_0; \\ n, & \mu = n - i_0 + 1, ..., n-1; \end{cases} \tag{8}$$

with i_0 and i_1 defined by

$$i_0 := [m/2] + 1, \qquad i_1 := m - [m/2]; \tag{9}$$

where, for any $t \in \mathbf{R}$, $[t] :=$ the maximum integer less than or equal to t.

The relevant values on $[a, b]$ of the set $\{w_i;\ i = 0, 1, ..., n\}$ can be evaluated from the formulas

$$w_i(x) := \begin{cases} \displaystyle\prod_{k=0,\, k \neq i}^{m-1} \frac{x - \tau_k}{\tau_i - \tau_k}, & x \in [a, \tau_{i_1 - 1}],\ (i \leqslant m - 1); \\ s_i(x), & x \in [\tau_{i_1 - 1}, \tau_{n - i_0 + 1}],\ (n \geqslant m); \\ \displaystyle\prod_{k=0,\, k \neq n-i}^{m-1} \frac{x - \tau_{n-k}}{\tau_i - \tau_{n-k}}, & x \in [\tau_{n - i_0 + 1}, b],\ (i \geqslant n - (m-1)); \end{cases} \tag{10}$$

where

$$s_i(x) := \sum_{r=0}^{m-2} \sum_{j=j_0}^{j_1} \alpha_{i,r,j} B^m_{(m-1)(i+j)+r}(x), \tag{11}$$

with

$$j_0 := \max\{-i_0, i_1 - 2 - i\}, \; j_1 := \min\{-i_0 + m - 1, n - i_0 - i\}.$$

The coefficients $\alpha_{i,r,j}$ are given in paper by De Villiers [3, Eq. (17)]. The B-splines series in (11) is constructed from the set $\{B_i^m;\ i=(m-1)(i_1-2), (m-1)(i_1-2)+1, ..., (m-1)(n-i_0+1)-1\}$ of normalized B-splines as defined in [12, p. 124]. We recall that

$$0 < B_i^k(x) \leqslant 1, \qquad \text{for} \quad x \in (x_i, x_{i+k}) \qquad \text{and} \qquad B_i^k(x) = 0 \text{ otherwise.} \tag{12}$$

The following uniform bounds hold [3, Theorem 4.2]

$$|\alpha_{i,r,j}| \leqslant \left[\sum_{\lambda=1}^{i_0} (R_n)^\lambda \right]^{m-1}, \tag{13}$$

and [3, Theorem 5.1]

$$|w_i(x)| \leqslant \left[\sum_{\lambda=1}^{m-1} (R_n)^\lambda \right]^{m-1}, \qquad x \in [a, b], i = 0, 1, ..., n. \tag{14}$$

We denote by $\| W_n \|$ the operator norm

$$\| W_n \| := \sup\{ \| W_n f \|_\infty;\ f \in C[a, b],\ \|f\|_\infty \leqslant 1\};$$

where $\|g\|_\infty := \max_{x \in [a,b]} |g(x)|$ is the maximum norm of a function $g \in \mathbf{C}[a, b]$, and by $\omega(g; \delta; I)$ the modulus of continuity

$$\omega(g; \delta; I) := \max_{\substack{x,\, x+h \in I \\ 0 < h \leqslant \delta}} |g(x+h) - g(x)|.$$

The operator norm $\| W_n \|$ is bounded by [6, Eq. (1.14)]

$$\| W_n \| \leqslant (m+1) \left[\sum_{\lambda=1}^{m-1} (R_n)^\lambda \right]^{m-1}, \tag{15}$$

and the following error estimates hold [6, Eqs. (1.16) and (1.17)]

THEOREM 2.1.

$$\|f - W_n f\|_\infty \leqslant \begin{cases} \|W_n\| \, \omega(f; mH_n; [a,b]), & \text{for } f \in C[a,b]; \\ (1 + \|W_n\|) \, c_{m,\nu} H_n^\nu \|D^\nu f\|_\infty & \text{for } f \in C^\nu[a,b], \ \nu = 1, 2, ..., m; \end{cases}$$

with the positive numbers $c_{m,\nu}$ *bounded by*

$$c_{m,\nu} \leqslant \begin{cases} \dfrac{(\pi m)^\nu \, (m-\nu)!}{2^{2\nu} m!}, & \text{for } \nu = 1, 2, ..., m-1; \\ \dfrac{m^m}{2^{2m-1} m!}, & \text{for } \nu = m. \end{cases}$$ ▮

From Theorem 2.1 and (15) the following sufficient conditions for uniform convergence can be deduced

COROLLARY 2.1. *Assume* $f \in \mathbf{C}[a,b]$. *Suppose that*

$$\{\Pi_n\} \text{ is locally uniform,} \tag{16}$$

and that

$$H_n \to 0 \quad \text{as} \quad n \to \infty. \tag{17}$$

Then

$$\|f - W_n f\|_\infty \to 0, \quad \text{as} \quad n \to \infty.$$ ▮

Moreover, the associated error sequence is $O(H_n{}^\nu)$, $\nu = 1, ..., m$, for correspondingly smooth f. A notable feature of the above mentioned estimates is that the bounds involved are independent of the placement of the knots of the secondary partition $X_n \backslash \Pi_n$.

3. SMOOTHNESS OF THE INTERPOLATION OPERATOR W_n

In order to continue the study on how well the nodal spline $W_n f$ approximates a smooth function, we introduce the following quantities [7, Eq. (4.1)]

$$E_{\nu,s}(x) := \begin{cases} D^\nu(f - W_n f)(x), & 0 \leqslant \nu < s; \\ D^\nu W_n f(x), & s \leqslant \nu < m; \end{cases}$$

where s is an integer with $1 \leqslant s \leqslant m$.

For $W_n f$ given by (6), we have

$$D^\nu W_n f(x) := \sum_{i=p_\mu}^{q_\mu} f(\tau_i)\, D^\nu w_i(x), \qquad x \in [\tau_\mu, \tau_{\mu+1}]. \tag{18}$$

By (10) and (11), $D^\nu w_i(x)$ involves the derivatives of the normalized B-splines $B_i^m(x)$. A bound for these derivatives is given by the following lemma, which can be quoted as a special case of [7, Lemma 2.1]

LEMMA 3.1. *Let $B_i^m(x)$ be the normalized B-spline defined over $x_i < x_{i+1} < \cdots < x_{i+m}$. Suppose $x \in [x_l, x_{l+1}]$ and $i \leqslant l < i+m$. Fix $0 < \nu < m$. Then $D^\nu B_i^m(x)$ exists, and*

$$|D^\nu B_i^m(x)| \leqslant \frac{\Gamma_{m,\nu}}{\delta_{i,l,m-1} \cdots \delta_{i,l,m-\nu}}, \tag{19}$$

where, for $j = m-\nu, ..., m-1$, we define $\delta_{i,l,j}$ as the minimum of $x_{r+j} - x_r$ with r such that $x_i \leqslant x_r \leqslant x_l < x_{l+1} \leqslant x_{r+j} \leqslant x_{i+m}$ and where

$$\Gamma_{m,\nu} := \frac{(m-1)!}{(m-\nu-1)!} \binom{\nu}{[\nu/2]}.$$

We denote by I_μ the interval

$$I_\mu := [\tau_{p_\mu}, \tau_{q_\mu}] = [x_{(m-1)p_\mu}, x_{(m-1)q_\mu}], \tag{20}$$

with p_μ and q_μ defined by (7) and (8). We will also need the following parameters describing the spacing of the partitions X_n and Π_n. Let

$$\Delta_\mu := \max_{(m-1)p_\mu \leqslant i \leqslant (m-1)q_\mu - 1} (x_{i+1} - x_i); \tag{21}$$

$$\mathscr{H}_\mu := \max_{p_\mu \leqslant i \leqslant q_\mu - 1} (\tau_{i+1} - \tau_i); \tag{22}$$

$$\bar{H}_n := \min_{0 \leqslant \mu \leqslant n-1} \mathscr{H}_\mu; \tag{23}$$

$$h_\mu := \min_{p_\mu \leqslant i \leqslant q_\mu - 1} (\tau_{i+1} - \tau_i). \tag{24}$$

To estimate $|E_{\nu,s}|$ we proceed as in [7]. Hence we state the following lemma [7, Lemma 4.1]:

LEMMA 3.2. *Suppose W_n is defined on a class of functions containing $\mathbf{P}^m$ and suppose (5) holds (i.e. W_n reproduces $\mathbf{P}^m$). Then for any polynomial $g \in \mathbf{P}^s$ and any f such that $D^v f(t)$ exists, where $0 \leqslant v < s \leqslant m$*

$$E_{v,s}(t) = \begin{cases} D^v(R(t) - W_n R(t)), & 0 \leqslant v < s \\ D^v W_n R(t), & s \leqslant v < m \end{cases} \tag{25}$$

with $R(x) := f(x) - g(x)$.

(a) *Local estimates.*

The purpose of this section is to obtain local estimates for $|E_{v,s}(t)|$, with $t \in [x_{l_\mu}, x_{l_\mu+1}] \subset [\tau_\mu, \tau_{\mu+1}]$, where $l_\mu \in \{(m-1)\mu, ..., (m-1)(\mu+1)-1\}$. Henceforth, for the rest of (a), we fix $\mu \in \{0, ..., n-1\}$.

We define for any $x \in I_\mu$ and $f \in C^{s-1}(I_\mu)$, with $1 \leqslant s \leqslant m$,

$$R(x) := f(x) - \sum_{i=0}^{s-1} \frac{f^{(i)}(t)}{i!} (x-t)^i. \tag{26}$$

Defining $R(x)$ by (26), so that $g(x)$ is the Taylor expansion of f at t, then $R(x)$ and its first $s-1$ derivatives are 0 at t. Hence, by (25), to give a bound to $|E_{v,s}(t)|$ it is only necessary to estimate $|D^v W_n R(t)|$. By using (18) we write

$$|D^v W_n R(t)| \leqslant \sum_{i=p_\mu}^{q_\mu} |R(\tau_i)| \, |D^v w_i(t)|. \tag{27}$$

We first estimate $|R(\tau_i)|$:

LEMMA 3.3. *Let $f \in C^{s-1}(I_\mu)$ with $1 \leqslant s \leqslant m$. Then for $i = p_\mu, ..., q_\mu$*

$$|R(\tau_i)| \leqslant \frac{m^s}{(s-1)!} \mathscr{H}_\mu^{s-1} \omega(D^{s-1}f; \mathscr{H}_\mu; I_\mu). \tag{28}$$

Proof. By the Taylor series for R we have

$$R(\tau_i) = \frac{D^{s-1}R(\eta_i)}{(s-1)!} (\tau_i - t)^{s-1}, \tag{29}$$

with η_i between τ_i and t, since $R(t) = \cdots = R^{(s-2)}(t) = 0$.

Since $q_\mu - p_\mu \leqslant m$, it follows that

$$|\eta_i - t| \leqslant |\tau_i - t| \leqslant \tau_{q_\mu} - \tau_{p_\mu} \leqslant m\mathscr{H}_\mu. \tag{30}$$

Using the subadditivity of $\omega(g; \delta; I)$ we derive from (26) and (30)

$$|D^{s-1}R(\eta_i)| = |D^{s-1}f(\eta_i) - D^{s-1}f(t)| \leqslant m\omega(D^{s-1}f; \mathscr{H}_\mu; I_\mu). \tag{31}$$

Now (29)–(31) yield the result. ∎

Let

$$\delta_{l_\mu, m-\nu} := \min_{l_\mu+1-m+\nu \leqslant r \leqslant l_\mu} (x_{r+m-\nu} - x_r), \; \nu = 0, 1, ..., m-1. \tag{32}$$

The following lemma estimates $|D^\nu w_i(t)|$

LEMMA 3.4. *Suppose* $t \in [x_{l_\mu}, x_{l_\mu+1}] \subset [\tau_\mu, \tau_{\mu+1}]$. *Then for* $\nu = 0, 1, ..., m-1$

$$|D^\nu w_i(t)| \leqslant \begin{cases} [(m-1)\cdots(m-\nu)] \left[\sum_{\lambda=1}^{m-1} (R_n)^\lambda\right]^{m-\nu-1} h_\mu^{-\nu}, \\ \qquad \text{for} \quad 0 \leqslant \mu \leqslant i_1 - 2 \quad \text{and} \quad n - i_0 + 1 \leqslant \mu \leqslant n-1; \\ m_\nu \left[\sum_{\lambda=1}^{i_0} (R_n)^\lambda\right]^{m-1} \dfrac{\Gamma_{m,\nu}}{(\delta_{l_\mu, m-\nu})^\nu}, \\ \qquad \text{for} \quad i_1 - 1 \leqslant \mu \leqslant n - i_0; \end{cases} \tag{33}$$

where

$$m_\nu := \begin{cases} 1, & \text{for} \quad \nu = 0; \\ m, & \text{for} \quad \nu > 0. \end{cases}$$

Proof. We consider the following cases.

I. Setting say $0 \leqslant \mu \leqslant i_1 - 2$, by (10) we have for $\nu = 1, ..., m-1$

$$|D^\nu w_i(t)| \leqslant \sum_{\substack{\rho_1=0 \\ \rho_1 \neq i}}^{m-1} \sum_{\substack{\rho_2=0 \\ \rho_2 \notin \{i, \rho_1\}}}^{m-1} \cdots \sum_{\substack{\rho_\nu=0 \\ \rho_\nu \notin \{i, \rho_1, ..., \rho_{\nu-1}\}}}^{m-1} \frac{1}{|\tau_i - \tau_{\rho_1}| \cdots |\tau_i - \tau_{\rho_\nu}|} \prod_{\substack{k=0 \\ k \notin \{i, \rho_1, ..., \rho_\nu\}}}^{m-1} \frac{|t-\tau_k|}{|\tau_i - \tau_k|}.$$

To obtain the first bound in (33) we use the inequality, following from inequalities (40) and (41) in [3], for $i \neq k$ and $a \leqslant x \leqslant \tau_{i_1-1}$,

$$\frac{|x-\tau_k|}{|\tau_i - \tau_k|} \leqslant \sum_{\lambda=1}^{m-1} (R_n)^\lambda,$$

and we take in account that

$$|\tau_i - \tau_{\rho_k}| \geqslant h_\mu \quad , \quad k = 1, ..., \nu,$$

since both τ_i and τ_{ρ_k} belong to I_μ.

We can proceed similarly for $n-i_0+1 \leqslant \mu \leqslant n-1$.

II. Let $i_1-1 \leqslant \mu \leqslant n-i_0$. We observe, directly from [4, Eq. (3.7)], (13) as well as (12), that,

$$|D^\nu s_i(t)| \leqslant \left[\sum_{\lambda=1}^{i_0} (R_n)^\lambda \right]^{m-1} \sum_{k=\lambda(i)}^{\sigma(i)} |D^\nu B_k^m(t)|, \tag{34}$$

where

$$\lambda(i) := \max\{(m-1)(i-i_0), l_\mu - m+1\},$$
$$\sigma(i) := \min\{(m-1)(i+i_1)-m, l_\mu\}.$$

The last sum in (34) is a sum of at most m terms, since, by (12), at most m B-splines are different from zero at t. By using (19) we have for $\nu = 1, ..., m-1$

$$\sum_{k=\lambda(i)}^{\sigma(i)} |D^\nu B_k^m(t)| \leqslant \Gamma_{m,\nu} \sum_{k=\lambda(i)}^{\sigma(i)} \frac{1}{\delta_{k,l_\mu,m-1} \cdots \delta_{k,l_\mu,m-\nu}} \leqslant m \frac{\Gamma_{m,\nu}}{(\delta_{l_\mu,m-\nu})^\nu}, \tag{35}$$

since, for $k=\lambda(i), ..., \sigma(i)$, $\delta_{k,l_\mu,j} \geqslant \delta_{l_\mu,j}$. By (12) and since

$$\sum_{k=l_\mu-m+1}^{l_\mu} B_k^m(t) = 1,$$

we note that, for $\nu=0$, the first sum in (35) is less than or equal to one.

Now (34) yields our result. ∎

Now we can give a local estimate for $|E_{\nu,s}(t)|$.

THEOREM 3.1. *Suppose* $t \in [x_{l_\mu}, x_{l_\mu+1}] \subset [\tau_\mu, \tau_{\mu+1}]$ *and let* $f \in C^{s-1}(I_\mu)$ *with* $1 \leqslant s \leqslant m$. *Then for* $0 \leqslant \nu < m$

$$|E_{\nu,s}(t)| \leqslant \begin{cases} K_1 \mathscr{H}_\mu^{s-\nu-1} \omega(D^{s-1}f; \mathscr{H}_\mu; I_\mu), \\ \qquad \text{for} \quad 0 \leqslant \mu \leqslant i_1 - 2 \text{ and } n-i_0+1 \leqslant \mu \leqslant n-1; \\ K_{2,\mu} \mathscr{H}_\mu^{s-\nu-1} \omega(D^{s-1}f; \mathscr{H}_\mu; I_\mu), \\ \qquad \text{for} \quad i_1 - 1 \leqslant \mu \leqslant n-i_0; \end{cases} \tag{36}$$

where

$$K_1 := \frac{m^{s+1}}{(s-1)!} [(m-1)\cdots(m-\nu)] \left[\sum_{\lambda=1}^{m-1} (R_n)^\lambda \right]^{m-\nu-1} (R_n)^{\nu(m-2)} \tag{37}$$

and

$$K_{2,\mu} := (m+1)\left[\sum_{\lambda=1}^{i_0} (R_n)^\lambda\right]^{m-1} m_\nu \frac{m^s}{(s-1)!}\, \Gamma_{m,\nu}\left(\frac{\mathscr{H}_\mu}{\delta_{l_\mu, m-\nu}}\right)^\nu. \tag{38}$$

Proof. We consider the following cases.

I. Let $0 \leqslant \mu \leqslant i_1 - 2$. The first inequality in (36) follows from (27), (28) and (33) since

$$h_\mu \geqslant (R_n)^{-(m-2)} \mathscr{H}_\mu,$$

and $q_\mu - p_\mu = m-1$. A similar procedure holds for $n - i_0 + 1 \leqslant \mu \leqslant n-1$.

II. Let $i_1 - 1 \leqslant \mu \leqslant n - i_0$. The second inequality in (36) follows from (27), (28) and (33) since $q_\mu - p_\mu = m$. ▮

(b) *Uniform bounds.*

The following uniform bounds can be deduced from the local estimates of Theorem 3.1.

Corollary 3.1. *Let $f \in C^{s-1}[a,b]$, with $1 \leqslant s \leqslant m$. Then, for $0 \leqslant \nu < s$*

$$\|E_{\nu,s}\|_\infty \leqslant K H_n^{s-\nu-1} \omega(D^{s-1}f; H_n; [a,b]) \tag{39}$$

and, for $s \leqslant \nu < m$,

$$\|E_{\nu,s}\|_\infty \leqslant K \bar{H}_n^{s-\nu-1} \omega(D^{s-1}f; H_n; [a,b]), \tag{40}$$

where

$$K := \max(K_1, K_2)$$

with K_1 defined by (37) *and*

$$K_2 := (m+1)\left[\sum_{\lambda=1}^{i_0} (R_n)^\lambda\right]^{m-1} m_\nu \frac{m^s}{(s-1)!}\, \Gamma_{m,\nu} (m-1)^\nu\, (r_n)^{\nu[i_0(m-1)-1]}, \tag{41}$$

with r_n defined by (1).

Proof. By the definitions in (21), (22) and (32) we have

$$\delta_{l_\mu, m-\nu} \geqslant x_{l_\mu+1} - x_{l_\mu}, \tag{42}$$

$$\mathscr{H}_\mu \leqslant (m-1)\, \Delta_\mu \leqslant (m-1)(r_n)^{i_0(m-1)-1}\, (x_{l_\mu+1} - x_{l_\mu}), \tag{43}$$

since, by (9), $i_0 \geqslant i_1$. From Theorem 3.1, by inserting (42) and (43) into (38), we get the uniform bounds (39) and (40) since, by the definitions (3) and (23),

$$\bar{H}_n \leqslant \mathscr{H}_\mu \leqslant H_n. \quad \blacksquare$$

For $f \in C^{s-1}[a, b]$, with $1 \leqslant s < m$, (40) provides a uniform bound of $|D^j W_n f|$ for $s \leqslant j < m$. In order to derive the sufficient conditions for uniform convergence, we need the following lemma:

LEMMA 3.5.

$$R_n \leqslant \sum_{\lambda=1}^{m-1} (r_n)^\lambda.$$

Proof. For all i and setting $j = i+1$

$$\tau_i - \tau_{i-1} = \sum_{\lambda=0}^{m-2} (x_{(m-1)i-\lambda} - x_{(m-1)i-\lambda-1})$$
$$\leqslant (x_{(m-1)i} - x_{(m-1)i-1}) \sum_{\lambda=0}^{m-2} (r_n)^\lambda$$

and

$$\tau_j - \tau_{j-1} \geqslant x_{(m-1)i+1} - x_{(m-1)i}$$
$$\geqslant (r_n)^{-1} (x_{(m-1)i} - x_{(m-1)i-1}). \quad \blacksquare$$

(c) *Uniform convergence.*

Now we can state the following corollary:

COROLLARY 3.2. *Assume that* $f \in C^{s-1}[a, b]$, *with* $1 \leqslant s \leqslant m$, *and* (17) *holds. Suppose that*

$$\{X_n\} \text{ is locally uniform.} \tag{44}$$

Then for $1 \leqslant v < s$,

$$\|D^v(f - W_n f)\|_\infty \to 0 \qquad as \quad n \to \infty. \tag{45}$$

Proof. The thesis follows immediately from Corollary 3.1, with $1 \leqslant v < s$, and Lemma 3.5. ∎

Finally, to improve the approximation power of the sequence $\{W_n f\}$, it would be reasonable to choose the secondary knots equally spaced throughout $[\tau_\mu, \tau_{\mu+1}]$ for all μ. Although the order of convergence is unchanged, we can state the following corollary:

COROLLARY 3.3. *Let $f \in C^{s-1}[a, b]$ with $1 \leqslant s \leqslant m$. Assume that the knots of the secondary partition $X_n \backslash \Pi_n$ are equally spaced throughout $[\tau_\mu, \tau_{\mu+1}]$, with $\mu = 0, ..., n-1$. Then*

(i) (39) *and* (40) *hold with* $K := \max(K_1, K_2^*)$ *where*

$$K_2^* := (m+1) \left[\sum_{\lambda=1}^{i_0} (R_n)^\lambda \right]^{m-1} m_v \frac{m^s}{(s-1)!} \Gamma_{m, v}(m-1)^v (R_n)^{v(i_0-1)}.$$

(ii) *If* (16) *and* (17) *hold, then* (45) *is true for* $1 \leqslant v < s$.

Proof. The proof of (i) is the same as Corollary 3.1, taking into account that

$$x_{l_\mu+1} - x_{l_\mu} = \frac{\tau_{\mu+1} - \tau_\mu}{m-1}$$

and

$$\mathscr{H}_\mu \leqslant (R_n)^{i_0-1} (\tau_{\mu+1} - \tau_\mu).$$

Property (ii) follows immediately from (41) since $r_n = R_n$. ∎

4. AN APPLICATION IN NUMERICAL INTEGRATION

We use the sequence $\{D^v W_n f\}$, with $0 \leqslant v < m-1$, for the numerical evaluation of CPV integrals of the form

$$J(w_{\alpha, \beta} D^v f; \lambda) := ⨍_{-1}^{1} w_{\alpha, \beta}(x) \frac{D^v f(x)}{x - \lambda} dx, \ \lambda \in (-1, 1) =: \mathring{J}; \qquad (46)$$

where

$$⨍_{-1}^{1} := \lim_{\varepsilon \to 0} \left\{ \int_{-1}^{\lambda - \varepsilon} + \int_{\lambda+\varepsilon}^{1} \right\};$$

$$w_{\alpha, \beta}(x) := (1-x)^\alpha (1+x)^\beta, \qquad \text{with} \quad \alpha, \beta > -1.$$

CPV integrals of the type (46) are evaluated numerically in [9] by using splines based on uniform and quasi uniform meshes.

Setting $J := [-1, 1]$, we define, for $\rho \in (0, 1]$, $\mathbf{H}_\rho(J) := \{g : \omega(g;t;J) \leqslant Bt^\rho, B > 0\}$. To prove the uniform convergence of $J(w_{\alpha,\beta} D^\nu W_n f; \lambda)$ to $J(w_{\alpha,\beta} D^\nu f; \lambda)$ as $n \to \infty$, we need the following convergence result [11, Theorem 5].

THEOREM 4.1. *Assume that $f \in H_\rho(J)$ for a given exponent $\rho \in (0, 1]$, and that we are given a sequence of functions $\{f_n\}$ such that*

1. $\|e_n\|_\infty = o(1)$, *where* $e_n := f - f_n$;
2. $e_n(1) = 0$ *if* $\alpha \leqslant 0$, $e_n(-1) = 0$ *if* $\beta \leqslant 0$;
3. $e_n \in \mathbf{H}_\sigma(J)$, $0 < \sigma \leqslant \rho$, *uniformly in n.*

Then

$$J(w_{\alpha,\beta} e_n; \lambda) \to 0 \quad as \quad n \to \infty, \text{ uniformly in } \lambda \in \mathring{J}, \tag{47}$$

if $\sigma + \min(\alpha, \beta, 0) > 0$.

We denote by $\mathcal{I}_{l_\mu}$ the interval

$$\mathcal{I}_{l_\mu} := [x_{l_\mu - m + 1}, x_{l_\mu + m}]$$

and we set

$$\bar{\Delta}_{l_\mu} := \max_{l_\mu - m + 1 \leqslant i \leqslant l_\mu + m - 1} (x_{i+1} - x_i).$$

We state the following theorem, proved in [11, Theorem 4].

THEOREM 4.2. *Let $0 \leqslant \nu < m - 1$, $f \in \mathbf{C}^\nu(J)$ and consider any sequence of locally uniform spline spaces $\{\mathbf{S}_{m,n}\}$. If any spline $S \in S_{m,n}$ satisfies*

1. $S \in \mathbf{C}^\nu(J)$;
2. $|D^\nu(f(t) - S(t))| \leqslant C_1 \omega(f^{(\nu)}; \bar{\Delta}_{l_\mu}; \mathcal{I}_{l_\mu})$, $x_{l_\mu} \leqslant t \leqslant x_{l_\mu + 1}$;
3. $|D^{\nu+1} S(t)| \leqslant C_2 \bar{\Delta}_{l_\mu}^{-1} \omega(f^{(\nu)}; \bar{\Delta}_{l_\mu}; \mathcal{I}_{l_\mu})$, $x_{l_\mu} < t < x_{l_\mu + 1}$.

Then

$$\omega(S^{(\nu)}; \Delta; J) \leqslant C_3 \omega(f^{(\nu)}; \Delta; J). \tag{48}$$

We shall state and prove the following uniform convergence theorem for CPV integrals of the form (46):

THEOREM 4.3. *Assume that (17) holds and that $f \in \mathbf{C}^\nu(J)$, with $0 \leqslant \nu < m - 1$, and $f^{(\nu)} \in \mathbf{H}_\rho(J)$ for a given exponent $\rho \in (0, 1]$. Suppose that (16) holds for $\nu = 0$ and (44) holds for $\nu > 0$. If $\nu = 0$ and $\rho + \min(\alpha, \beta, 0) > 0$, or if $\nu > 0$ and $\alpha, \beta > 0$, then (47) holds with $e_n = D^\nu(f - W_n f)$.*

Proof. Theorem 4.1. 1 is true by Corollary 2.1, for $v=0$, and by Corollary 3.2, for $v>0$.

Theorem 4.1. 2 holds for $v=0$ and is not required for $v>0$, since $\alpha, \beta>0$.

By the assumption $\rho+\min(\alpha, \beta, 0)>0$ if $v=0$ and, if $v>0$, by observing that, since $\alpha, \beta>0$, $\sigma+\min(\alpha, \beta, 0)>0$ for any $\sigma>0$, we need to verify Theorem 4.1, 3 with $\sigma=\rho$. To do this we have to prove that $W_n^{(v)}f \in \mathbf{H}_\rho(J)$, since, for any $u, v \in J$,

$$|e_n(u)-e_n(v)| \leqslant |f^{(v)}(u)-f^{(v)}(v)|+|W_n^{(v)}f(u)-W_n^{(v)}f(v)|.$$

By observing that $\mathscr{I}_{l_\mu} \subset I_\mu$ and

$$x_{l_\mu+1}-x_{l_\mu} \leqslant \bar{\Delta}_{l_\mu},$$

we can derive, from (36)–(38), (42), (43) and by subadditivity of the modulus of continuity, that $W_n f$ satisfies the conditions required by Theorem 4.2. Now (48) yields our result. ∎

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